Anti-magic and Edge Graceful Graphs

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Abstract

An Anti-Magic labeling for a graph $G = \{V, E\}$ is a bijection between the numbers $\{1, ..., |E|\}$ and the edges such that the sum of all edges connected to a vertex is distinct. in 1990 Hartsfield and Ringel conjectured that all simple connected graphs except K_2 have an Anti-Magic labeling. We will discuss and summarize recent results in the field of edge labeling for graphs, specifically Anti-Magic graphs and their variations.

1 Introduction

We will be talking about edge labeling for graphs, specifically Anti-Magic graphs variants. Anti-Magic graphs were introduced in 1990 by Hartsfile and Ringel[1]. The variants we will be talking about are Edge Graceful, k-Anti-Magic, (ω, k)-Anti-Magic, and Oriented Anti-Magic.

Firsts we will define an Anti-Magic labeling.

Definition 1.1. An Anti -agic graph is a graph with an Anti-Magic labeling of the edges. An Anti-Magic labeling of a graph with m edges and n vertices is a bijection from the set of edges to the integers $\{1, ..., m\}$ such that all n vertex sums are pairwise distinct. A vertex sum is the sum of labels of all edges incident with that vertex.

In 1990, Hartsfield and Ringel conjectured that every simple connected graph, other than K_2, is Anti-Magic. We can see via inspection that K_2 is not Anti-Magic. Whether this conjecture is true remains an open problem today.[2] From now on every graph in this paper can be assumed to be simple, which means they have no double edges or self loops.

In this paper we will summarize current progress to proving graphs are Anti-Magic. We will show how close we are to proving different kinds of graphs are Anti-Magic. We will start Edge Graceful graphs and continue with the weaker forms of k-Anti-Magic, and (ω, k)-Anti-Magic, the finish with Oriented Anti-Magic.

1.1 Variants of Anti-Magic Graphs

We will talk about several different variants of Anti-Magic graphs.

Definition 1.2. An Edge Graceful graph is an Anti-Magic graph where the vertex sums are distinct modulo $|V|$.

Definition 1.3. A k-Anti-Magic labeling for a graph for non-negative integer k is an injection from the set of edges to the integers $\{1, 2, ..., m+k\}$ such that all n vertex sums are pairwise distinct.

We see that a 0-Anti-Magic Graph is an Anti-Magic graph.

Definition 1.4. A (ω, k)-Anti-Magic labeling, with k a non negative integer and ω a weight function of the set of vertices, is an injection from the set of edges to the integers $\{1, 2, ..., m+k\}$ such that all n vertex sums are pairwise distinct, where a vertex sum is the sum of labels of all edges incident with that vertex and its initial weight under ω.

We see that a (0,0)-Anti-Magic graph is an Anti-Magic graph, where the first 0 is the 0-function which maps all the vertices to 0 and the second 0 is just the number 0. And that a $(0, k)$-Anti-Magic graph is a k-Anti-Magic graph.

Definition 1.5. An oriented Anti-Magic graph is a version of Anti-Magic graphs for directed graphs. An oriented Anti-Magic labeling of a digraph with m edges and n vertices is a bijection from the set of edges to the integers $\{1, 2, ..., m\}$ such that all n oriented vertex sums are pairwise distinct, where an oriented vertex sum is the sum of labels of the edges entering that vertex minus the sum of labels of the edges leaving it.

There are 2 general ways people go about trying to push towards proving the Anti-Magic conjecture. The first is to build up types of graphs and combinations of those that we know are Anti-Magic; the other is to use these other forms of Anti-Magic graphs and try to lower the bound on k and ω until it can be shown that they both can be 0.

2 Edge Graceful Graphs

Edge Graceful graphs is the strongest form of labeling and thus we know the least about it. Here we just show a single theorem from Hefetz in 2005.[3]

Theorem 2.1. *Let $G = (V, E)$ be a graph on n vertices and m edges, such that $G = H \cup f_1 \cup ... \cup f_r$ where $H = (V, E')$ is edge graceful and the f_i's are 2-factors; then G is edge graceful.*

Definition: A k-factor of a graph is a sub-graph that includes all of the vertices and subset of the edges such that the subgraph is k-regular.

We note that any 2-factor must be comprised of 1 or more cycles in the graph, since every node has degree 2.

Proof. Let $G = H \cup f_1 \cup ... \cup f_r$ $|V(G)| = |V(H)| = n$ and let ω be an edge graceful labeling of H. We will prove this via induction on r. Our base case is $r = 0$. If $r = 0$ then $G = H$ and we have an edge graceful labeling on G since H is required to be edge graceful. Now we will show that if we have an edge graceful labeling of a graph, we can always add a 2-factor and get another edge graceful labeling. This means that for any $r > 0$ we can remove a 2-factor f_r from G. By the induction hypothesis, we know that $G' = G \backslash f_r$ is edge graceful.

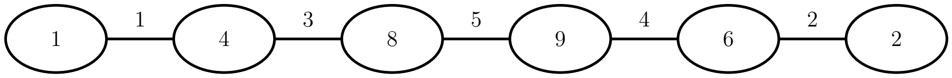

Figure 1: A path that is labeled as described in Theorem 3.1.

Let ω' be that edge graceful labeling.

We will now show how we can label the edges in this 2-factor such that the total graph is Edge Graceful. We will label these edges with the labels $|E'| + 1, ..., |E'| + n$ which are $0, 1, ..., n - 1$ modulo n.

We now look at each of the cycles in f_r such as $v_1 - e_1 - v_2 - e_2 - ... - v_k - e_k - v_1$ Let us call the vertex sums of these vertices in the Edge Graceful graph G' to be $a_1, ..., a_k$. We label each edge from f_r with a_i^{-1} in $\langle \mathbb{Z}_n, + \rangle$. Remember that our labels corresponded int the field $\mathbb{Z}_n$ to $0, 1, ..., n - 1$ and so we are guaranteed to have each label. Now for this cycle the vertex sum of each vertex i is the inverse in $\langle \mathbb{Z}_n, + \rangle$ of the vertex sum of vertex $(i - 1)$ mod n. Which we know are unique, which gives us a new Edge Graceful labeling. We now do this for every cycle in f_r and have a Edge Graceful labeling of G. □

3 Anti-Magic Graphs

We know several general kinds of graphs are Anti-Magic including P_n $(n \geq 3)$, cycles, wheels and K_n $(n \geq 3)$. Here we will give a quick proof for the first 3.

Theorem 3.1. *Every Path graph of length at least 3 is Anti-Magic. Where a Path graph is one for which the vertices can be listed in such an order that v_i has an edge to v_{i+1} for all $1 \leq i < n$.*

Proof. Assign an ordering to the vertexes such that v_1 and v_n are both of degree 1 and the edges are $e_i = (v_i, v_{i+1})$ for $1 \leq i < n - 1$. Label the edges from e_0 to $e_{\lceil \frac{n}{2} \rceil}$ with $2i + 1$ and label the edges from e_{n-1} to $e_{\lceil \frac{n}{2} \rceil + 1}$ with 2 through $\lfloor \frac{n}{2} \rfloor$ with the 2 on e_{n-1}.

This means that the vertex sums are as follows: v_1 is 1, v_i for $2 \leq i \leq \lfloor \frac{n}{2} \rfloor$ is $2i - 1 + 2i + 1 = i * 4$, $v_{\lfloor \frac{n}{2} \rfloor + 1}$ is $m + m - 1$, v_i for $\lfloor \frac{n}{2} \rfloor + 2 \leq i \leq n$ is $(n - i) * 4 + 2$. We note that the first one is uniquely one, the next set are increasing and $0 \mod 4$ the middle one is an odd number, and the last set are decreasing $2 \mod 4$, thus they are all unique. We see in the example in Figure 1 that v_1 is 1, v_2 and v_3 are 4 and 8 respectively, the middle is 9, the last count down with 6 and 2, both $2 \mod 4$. □

Theorem 3.2. *Every Cycle graph is Anti-Magic. Where a Path graph is one for which the vertices can be listed in such an order that v_i has an edge to $v_{i+1 \mod n}$ for all $1 \leq i \leq n$ and the number of vertices is at least 3.*

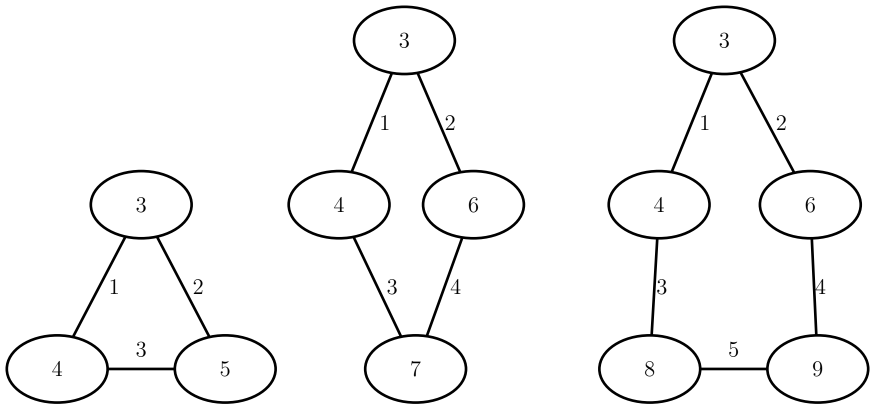

Figure 2: We see how we can always add a edge to a cycle graph as described in Theorem 3.2

Proof. Start with a cycle of length 3, label the edges with 1,2, and 3. The vertex sums are 3,4,and 5. Thus it is Anti-Magic, this can be seen in Figure 2. We will show how we can always take a vertex and add it to this graph and make another Anti-Magic graph. Start with the cycle of length 3, then take the edge labeled 3 and break it into 2 edges and add a vertex in the middle. Leave the label attached to the vertex with the smaller sum alone and increase the other label by 1 to 4. Now we have vertices with weights 3 and 4 that did not change, we increased 1 of the old vertices from 5 to 6, and the new vertex we added is 7. This is also shown in Figure 2. We note that the 2 nodes of max label are still next to each other. We do the same thing again. Split the edge of max weight keeping it with the same value along the side with the lower label. This means we take the old node of max weight and add 1 and make a new node that has weight higher than that, since it is connected to the 2 edges of maximum label, since no other are changed we are left with an Anti-Magic graph. Thus any cycle graph must be Anti-Magic. □

In 2005, Wang proved that the Cartesian product ($C_m \times C_n$) and higher dimensional variants ($C_{m_1} \times ... \times C_{m_t}$) are Anti-Magic. And moreover he showed that the Cartesian product of a cycle graph with any k-regular graph is Anti-Magic for $k \geq 2$.[4]

Theorem 3.3. *Every Wheel graph is Anti-Magic. Where a Wheel graph is a cycle graph with an extra node that has an edge connecting it to every other node.*

Proof. We use the proof of Theorem 3.2 and show that we can always add another node that connects to all other nodes. We start by ordering the vertices so that $1, ..., n-1$ are the cycle and n is in the middle. We order the edges so that $1, ..., n-1$ are in the cycle and $n, ..., 2n-2$ connect to the center node. First label all edges in the cycle with the labels $1, ..., n-1$ as we did before. Call this labeling ω. Label the edge going to the lowest weight in ω with the lowest remaining edge, which is $n+1$. Keep doing this. Each vertex going in order of orders in ω gets the next lowest remaining label. Since each bigger label gets a label bigger than everything else all vertex sums in the cycle must be unique. The vertex sum in the middle is more than the sum of the three biggest weights, so it must be different from the rest which all have degree three. This can be seen in Figure 3. □

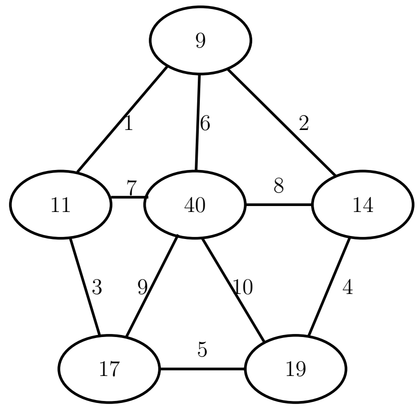

Figure 3: A Wheel graph as described in Theorem 3.3

We also know that complete graphs on 3 or more vertices are Anti-Magic. Moreover we know that every graph with high enough minimum degree is Anti-Magic. Alon showed in 2003 that

Theorem 3.4. *There exists an absolute constant C such that every graph with n vertices and minimum degree at least* $C * \frac{\log(n)}{\log(\log(n))}$ *is Anti-Magic.[5]*

4 k-Anti-Magic Graphs

We know that given enough labels we can always label a graph so we start with a trivial bound that all simple connected graphs with n vertices and m edges must be 2^m-Anti-Magic. This can easily be seen since if we label edge e_i with 2^i then we have all distinct vertex sums since their are no combinations of unique edges that collide.

In 2005 Hefetz used Alon's Nullstellensatz to give several new bounds for different types of graphs[3]. We start with one that is more general than simple connected but still manages to lower the bound to a constant factor of the number of nodes.

Theorem 4.1. *Every graph with at most one isolated vertex and no isolated edges is* $2|V|-4$*-antimagic. Furthermore it is* $(\omega, 2|V|-4)$*-Anti-Magic for every initial weight function* ω*.*

Proof. We will show how to construct such a labeling for any graph fitting the criteria. First assign an arbitrary ordering of the edges $\{e_1, ..., e_m\}$. At stage i, assign a weight to edge e_i from the set $\{1, 2, ..., m+2n-4\}$ such that the vertex sum of both endpoints differ from that of any other endpoints. We note that it is possible that the two end points of an edge have the same value, we ensure only that they differ from all others. In fact, for the first label they will always have the same value if the initial weight of the 2 edges was the same. At each stage, each endpoint can be assigned to anything but $n-2$ items,which are the values that would collide with that of any other vertex sum. Since each edge effects 2 vertexes, this gives us up to $2n-4$ forbidden values for each edge. However at each step we have $(m+2n-4)$ different options less the $(m-1)$ labels we might have already used giving us $2n-3$ possible labeling so we always have a possible label.

We note that at no stage did we require any information about the initial weight of the vertices so this same method works for any starting vertex weights.

□

Theorem 4.2. *If $G = (V, E)$, where $|V| > 2$, admits a 1-factor then is it $(|V|-2)$-antimagic*

To prove this we need a few other theorems.

Theorem 4.3 (Combinatorial Nullstellensatz)**.** *Let F be an arbitrary field, and let $f = f(x_1, ..., x_n)$ be a polynomial in $F[x_1, ..., x_n]$. Suppose the degree $deg(f)$ of f is $\sum_{i=1}^{n} t_i$, where each t_i is a nonnegative integer, and suppose the coefficient of $\prod_{i=1}^{n} x_i^{ti}$ in f is nonzero. Then, if $S_1, ..., S_n$ are subsets of F with $|S_i|>t_i$, there are $s_1 \in S_1, s_2 \in S_2, ..., s_n \in S_n$ so that $f(s_1, ..., s_n) \neq 0$. [6]*

We will also need a special Case of the Dyson Conjecture proved in the 1960s by many people including [7] and [8].

Lemma 4.4. *For every positive integers k, n let $c_{k,n}$ be the coefficient of $\prod_{i=1}^{n} x_i^{k(n-1)}$ in $V_n^{2k}(x_1, ..., x_n) = \prod_{n \geq i \geq j \geq 1}(x_i - x_j)^{2k}$; then $c_{k,n} \neq 0$*

Now we can do the proof of Theorem 4.2

Proof. Let G be a graph on $2n$ vertices, and $M = \{(u_i, v_i) : 1 \leq i \leq n\}$ a 1-factor of G. The $m - n$ edges of $G \backslash M$ can be labeled such that the vertex sum of u_i differs from the vertex sum of v_i using the integers $\{1, 2, ..., m - n + 2\}$ using the same scheme we used in 4.1. We notice that, in this case, for each vertex we only have 1 other vertex that it has to avoid, the single vertex that it is connected to in the 1-factor, which is why we can label the graph $G \backslash M$ with $m - n + 2$ labels.

For every vertex in G we denote its initial weight under this scheme as $\omega(v)$. Now we will show that we can label the edges of M. Let x_i be the label of edge (u_i, v_i) in M. We need the following to be true for all vertex sums to be different for $1 \leq i<j \leq n$:

$$x_i + \omega(u_i) \neq x_j + \omega(u_j)$$
$$x_i + \omega(u_i) \neq x_j + \omega(v_j)$$
$$x_i + \omega(v_i) \neq x_j + \omega(u_j)$$
$$x_i + \omega(v_i) \neq x_j + \omega(v_j)$$

Note: we also need $\omega(u_i) \neq \omega(v_i)$, but this is true by our initial labeling

So we just need to show that there exists a vector $\bar{x} = (x_1, ..., x_n)$ from the set of unused labels $\{1, ..., m + 2n - 2\}$ such that

$$P_M(\bar{x}) = \prod_{i<j}(x_i - x_j)*$$
$$(x_i - x_j + \omega(u_i) - \omega(u_j))*$$
$$(x_i - x_j + \omega(u_i) - \omega(v_j))*$$
$$(x_i - x_j + \omega(v_i) - \omega(u_j))*$$
$$(x_i - x_j + \omega(v_i) - \omega(v_j)) \neq 0$$

This is satisfied exactly when all 4 of the equations above can be satisfied.

Which is the same as finding a non-vanishing $\prod_{i=1}^{n} x_i^{t_i}$ in P_M such that $\sum_{i=1}^{n} t_i = deg(P_M)$ and $t_i < 3n-2$ for $1 \leq i \leq n$ by Theorem 4.3.

Which is the same as the monomial in $V_n^5(\bar{x}) = \prod_{i>j}(x_i - x_j)^5$, which is true by Lemma 4.4. □

Theorem 4.5. *If $G = (V, E)$ is a $(2d+1)$-regular bipartite graph, where $d \geq 1$, and if moreover there exists a decomposition of G into a 1-factor and d 2-factors whose every circuit is of length divisible by 4 then G is $\left(\frac{|V|}{2} - 1\right)$-antimagic*

Proof. Let $G = (U \cup V, E)$ be a bipartite graph with $U = \{u_1, ..., u_n\}$ and $V = \{v_1, ..., v_n\}$. And let $\{(u_i, v_i) | 1 \leq i \leq n\}$ be a 1-factor and let $\{f_i | 1 \leq i \leq d\}$ be 2 factors with no circuits of length 2 mod 4. We note that a circuit in a bipartite graph cannot be of odd length since for every time it goes from U to V it must also go back. This means that every circuit is of length $0 \mod 4$.

Label f_1 with $(1, 2dn, 2, 2dn-1, ..., n, 2dn+1-n)$ starting with an arbitrary edge and label the rest of the $f_i's$ with $(n(i-1)+1,\ 2dn - n(i-1),\ 2dn - n(i-1) - 1,\ n(i-1)+2,\ n(i-1)+3\ ,...,\ n(i-1)+n,\ 2dn - n(i-1) - n + 1)$ again starting with an arbitrary edge.

The sum of each vertex of V is $d(2dn+1)$ of which $2dn+1$ is from each 2 factor. Let the weight of each vertex v, after dealing with all the 2-factor graphs, be denoted $\omega(v)$.

Denote the edges of the 1-factor graph $\{(u_i, v_i) | 1 \leq i \leq n\}$ as $x_1, ..., x_n$ and we need the same 4 conditions as Theorem 3

$$x_i + \omega(u_i) \neq x_j + \omega(u_j)$$
$$x_i + \omega(u_i) \neq x_j + \omega(v_j)$$
$$x_i + \omega(v_i) \neq x_j + \omega(u_j)$$
$$x_i + \omega(v_i) \neq x_j + \omega(v_j)$$

Since we have $\omega(v_1) = ... = \omega(v_n)$ we can get rid of the extra constraint $x_i \neq x_j$ meaning we now only need a monomial of degree $<2n-1$, which is again guaranteed by the Lemma 4.4 □

Theorem 4.6. *Let G be a graph on n vertices and maximal degree $n-k$, where $k \geq 3$ is any function of n: then G is $(3k-7)$ antimagic*

Proof. Let $G=(V,E)$ where $|V|=n$ and $|E|=m$ be a graph of maximal degree $n-k, k \geq 3$. Let $v \in V$ be a vertex of maximal degree $(n-k)$ and $v_1,...,v_{n-k}$ be its neighbors. Let $A=\{u_1,...,u_{k-1}\}$ be the rest of the vertices in G. This partitions the vetices into 3 groups. We will now split up our edges into three groups $E=E_1 \cup E_2 \cup E_3$, by which vertices they are connected to. E_1 is the set of edges from v. E_2 is the set of edges with at least 1 endpoint in A, that is to say edges that touch a vertex that is not connected to the vertex of maximal degree. E_3 is the possibly empty subset of edges that connect two neighbors of v, that is to say go from v_i to v_j.

We now label the edges in E_2 as in Theorem 4.2. For edges contained in A we have $2k-6$ forbidden labels and for edges that have a v_i we have $k-2$ forbidden labels (for u_i). Thus we can do these labels using $|E_2|$ from the set $\{1,...,|E_2|+2k-6\}$. These vertex sums of $u_1,...,u_{k-1}$ as $a_1,...,a_{k-1}$. Note that these are final since no other edges touch these vertices.

Label E_3 arbitrarily using the smallest unused labels.

Denote the current vertex sums of v_i as $\omega(v_i)$ and assume without loss of generality that $\omega(v_1) \leq ... \leq \omega(v_{n-k})$. Now label the edges of E_1 with the largest $n-1$ labels $(b_1<...<b_{n-1})$. Note that we have only used the bottom $m+2k-6$ and we have $m+3k-7$.

We do this as follows, (v,v_1) is given the smallest integer b_{i_1} where $1 \leq i_1 \leq n-1$ is the smallest integer such that $\omega(v_1)+b_{i_1} \neq a_t$ for every $1 \leq t \leq k-1$, which must exist since we have $n-1$ labels and $k-1$ restrictions, and so on. After labeling (v,v_j) we have $n-1-j$ unlabeled edges, $n-1-i_j$ labels and at most $k-1+j-i_j$ restrictions giving us a labeling $\omega'(v_1)<...<\omega'(v_{n-k})$. □

Hefetz continues and shows that if $n \geq 6k^2$ then we can improve our bound to $(k-1)$-Anti-Magic[3]. He also shows that

Theorem 4.7. *Let G be a graph on $n=3^k$ vertices, $k \in \mathbb{N}$. if G admits a K_3-factor then G is Anti-Magic*

Hefets later generlizes this with Saluz and Tran in 2009 to the following. If G is a graph on $n=p^k$ vertices where p is an odd prime and k is an positive integer, and G admits a k-factor, then G is Anti-Magic.[9]

5 Oriented Anti-Magic

Theorem 5.1. *If $G=(V,E)$ is a $(2d+1)$-regular bipartite graph, where $d \geq 1$, G has an orientation such that the resulting digraph is Anti-Magic*

For the proof we follow similar to that of 4.5

Proof. We once again define our graph to be $G = (U \cup V, E)$ be a bipartite graph with $U = \{u_1, ..., u_n\}$ and $V = \{v_1, ..., v_n\}$. We start by directing all edges from U to V and remove a single perfect matching $\{(u_i, v_i) | 1 \leq i \leq n\}$, which we are guaranteed to have since it is a regular bipartite graph. And label the rest of the edges.

Now regardless of the weights of the last edges, we add the v_i that are positive and the u_i that are negative and so we only have a single condition

$$x_i + \omega(u_i) \neq x_j + \omega(u_j)$$

Which means by the same logic as before we only need to find a monomial of degree $<n$ which is again found by the Lemma 4.4 □

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